Topological edge states in Rashba-Dresselhaus spin-orbit-coupled atoms in a Zeeman lattice

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We address the impact of spin-orbit coupling on the existence and properties of topological edge states of cold neutral atoms and Bose-Einstein condensates loaded in honeycomb Zeeman lattices—lattices where the spinor components are placed in potentials having opposite signs. We find that the type of spin-orbit-coupling mechanism has profound effect on the emergence of topological edge states. We also reveal that edge states persist when interactions are present and that they become metastable in Bose-Einstein condensates.

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Discrete and continuous lattices exhibit degeneracies in the eigenmode spectrum when the corresponding Hamiltonian is characterized by suitable spatial symmetries and time-reversal invariance [1]. Graphene, as the paradigm of a honeycomb lattice [2], is one of the best-known examples of structures where energy bands touch at Dirac points. If the underlying symmetries are broken, a gap may open at the Dirac points thus leading to a transition to either a conventional or a topological insulator phase, depending on which symmetry is broken [1]. When such a lattice is located in contact with a material having distinct topological properties, topological states with energies falling into the gap and localized at the edge between two materials may appear. An outstanding perturbation leading to the appearance of topological edge states is spin-orbit coupling (SOC), which in electronic systems gives rise to the quantum spin Hall effect [3,4].

Interest in topological edge states is constantly growing [4,5] and to date the concept of topological insulation has been extended to several areas of physics, where SOC can be emulated by coupling the translational and the internal spinor degrees of freedom, the latter often referred to as pseudospin. Topological insulators have been realized in acoustic [6] and mechanical systems [7], as well as in optical and optoelectronic systems [8], including gyromagnetic photonic crystals [9-11], semiconductor quantum wells [12], arrays of coupled resonators [13,14], metamaterial superlattices [15], helical waveguide arrays [16–18], systems with driving fields containing vortex lattices [19], and in polaritonic systems, where SOC originating in splitting of energy levels for different polarization states leads to topological effects [20-22]. In atomic systems with SOC induced by an homogeneous field, the spin Hall effect was observed experimentally [23].

Simulation of quantum spin Hall effect with atoms in optical lattices subject to the field gradient was proposed in [24].

A synthetic SOC can be simulated in atomic systems [25–27], where proper superpositions of hyperfine atomic states, which are described by two- (or multi-) component wave functions, have spinor character, and therefore mimic spin. Such states are considered to bear a pseudospin. In atomic systems with the SOC induced by a homogeneous field, the spin Hall effect was observed experimentally [23].

In this Rapid Communication we show that topological edge states can be realized in systems of cold pseudo-spin-1/2 SOC atoms and SOC Bose-Einstein condensates (SO-BEC) [25,26] embedded in a Zeeman lattice. We consider periodically varying Zeeman splitting induced by external fields forming a lattice. Such lattices having opposite signs for two spinor components may obey a desirable symmetry [28] and are feasible experimentally [29]. The peculiarity of the Zeeman lattice is that when the spinor components in a Zeeman lattice are uncoupled, one of them is localized in deep potential wells and is well described by the tightbinding approximation, while the second component is in the "almost-free-electron" limit. SOC links spinor components, i.e., it couples these states with practically opposite dynamical properties. Tunability of SOC in atomic systems [26], where nearly arbitrary gauge potentials can be created [27], affords the exploration of the interplay of the Rashba and Dresselhaus mechanisms in the formation of topological edge states. When considering SO-BEC, the nonlinearity stemming from twobody interactions becomes relevant [30]. Since the scattering length can be of any sign, one can explore topological states in different nonlinear regimes. We present an example of the nonlinear topological edge states in atomic condensates and study their stability.

We address a SOC atom described by the spinor $\boldsymbol{\psi} = (\psi_1, \psi_2)^T$ (*T* stands for the transpose), whose

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FIG. 1. Upper panel: the Zeeman lattice with zigzag (left) and bearded (right) edges used in simulations (three unit cells are shown). Lower panels: examples of edge states for different Bloch momenta. Only $|\psi_1|$ is shown. In all cases $\beta_x = \beta_y = 1.5$.

evolution is governed by the Schrödinger equation $i \partial \psi / \partial t =$ $(\hat{H}_0 + \hat{H}_{\text{SOC}})\psi$. Here $\hat{H}_0 = (\hat{k}_x^2 + \hat{k}_y^2)/2 + \hat{U}(\mathbf{r}), \, \hat{k}_x = -i\partial_x,$ $\hat{k}_y = -i\partial_y$, and the units in which $m = \hbar = 1$ are used. The SOC is described by $\hat{H}_{SOC} = \beta_y \sigma_x \hat{k}_y - \beta_x \sigma_y \hat{k}_x$, where the positive constants $\beta_{x,y}$ are determined by the interplay between Rashba SOC and Dresselhaus SOC, and $\sigma_{x,y,z}$ are the Pauli matrices. The case $\beta_x = \beta_y = \beta_R$ corresponds to the pure Rashba SOC $\beta_{\rm R}(\sigma_x \hat{k}_y - \sigma_y \hat{k}_x)$ [31]. Similar gauge potentials were previously used in atomic systems [32], where they are created by the external coupling of hyperfine states. This crucial flexibility is a difference between atomic systems and their solid-state counterparts, where the particular form of SOC depends on the lattice symmetry [33]. An independently engineered Zeeman lattice [29] is created to have identical functional shapes, but opposite signs for two different spinor components. We model the Zeeman lattice by the potential $\hat{U}(\mathbf{r}) = -\mathcal{R}(x, y)\sigma_z$, where $\mathcal{R}(x, y) = \rho \sum_{m,n} e^{-[(x-x_m)^2 + (y-y_n)^2]/d^2}$ describes a honeycomb structure with the amplitude ρ , characteristic width d of lattice sites at the nodes (x_m, y_n) of the discrete honeycomb grid. The distance between neighboring sites is a. In these notations $-\mathcal{R}(x, y)$ and $+\mathcal{R}(x, y)$ are the potentials for the components ψ_1 and ψ_2 , respectively. Thus, domains where ψ_1 tends to have maxima are the regions with strongest expulsion for ψ_2 . This leads to nontrivial competition between the Zeeman lattice and SOC, since the latter tends to create nonzero density in the ψ_2 component in the vicinity of density maxima of the ψ_1 component, i.e., around maxima of potential for ψ_2 .

We assume that the lattice is infinite along the y axis and is truncated along the x axis. The truncation is such that the lattice has two different edges, where topological states can appear: zigzag and bearded ones (Fig. 1, top row). Such truncation allows us to compare directly the properties of the modes excited at the edges of different types. The modes of such truncated lattice are Bloch waves of the form $\psi(x, y) = e^{iky-i\varepsilon(k)t}\phi(x, y)$, where $\varepsilon(k)$ is the energy, $k \in [0, K]$ is the Bloch momentum along the y axis, ϕ is the periodic spinor function with a period $T = 3^{1/2}a$ along the y axis, and $K = 2\pi/T$ is the width of the Brillouin zone.

In the absence of SOC, i.e., when $\beta_x = \beta_y = 0$, the spinor component ψ_j (j = 1, 2) solves the stationary Schrödinger equation with the potential $(-1)^j \mathcal{R}(x, y)$. Since the potential

 $-\mathcal{R}(x, y) < 0$ represents an array of narrow wells, the ground state of the ψ_1 component has negative energy. The potential $+\mathcal{R}(x, y)$ is positive and the corresponding energy spectrum for ψ_2 is located at $\varepsilon > 0$. This peculiarity has several consequences. First, in the presence of weak SOC the opening of the lowest topological gap is expected at those (negative) energy levels, where degeneracies in the form of Dirac points are encountered in the spectrum of ψ_1 when it is decoupled from ψ_2 . Second, in this lowest topological gap all states are characterized by the dominating ψ_1 component. Finally, for a deep lattice, like the one used here within the framework of the full continuous model, the decoupled component ψ_1 is well described by the tight-binding approximation [34].

One of our central findings is that the type of SOC, or more precisely the relation between SOC strengths β_x and β_y is a decisive factor determining whether topological modes can be created. Indeed, let us first consider the case when one of the SOC components is nonzero, but small enough to be considered as a perturbation of the Hamiltonian \hat{H}_0 of the infinite lattice. Let \mathcal{T} be the time-reversal operator that changes $\hat{\mathbf{k}} \rightarrow -\hat{\mathbf{k}}$ and performs complex conjugation of the wave function: $\mathcal{T}\psi(\mathbf{r}) = \psi^*(\mathbf{r})$. The gap in the spectrum of the infinite lattice can open only in the vicinity of the Dirac points under the action of perturbations. However, for a pair of Dirac points with coordinates $[0, \pm 2K/3]$ in the reciprocal lattice, the SOC component $\beta_x \sigma_y k_x$, which does not break timereversal symmetry \mathcal{T} , becomes exactly zero, since in these points $k_x = 0$. Thus, a total gap cannot be opened by such a perturbation. On the other hand, perturbation introduced by another SOC component $\beta_{v}\sigma_{x}k_{v}$ that acquires nonzero value $\beta_v \sigma_r (2K/3)$ in the above-mentioned Dirac points, does open the gap. We have verified these properties numerically not only for small values of SOC strengths, but also for $\beta_x \sim 1$ and $\beta_y \sim 1$ (keeping $\beta_y = 0$ or $\beta_x = 0$, respectively). Even though $\beta_y \sigma_x \hat{k}_y$ perturbation opens the gap around Dirac points of \hat{H}_0 and even though it is neither \mathcal{T} - nor \mathcal{P} -symmetric (where \mathcal{P} is the operator of spatial inversion), it cannot lead to the appearance of topological states, when it acts alone. The reason behind this is that unperturbed Hamiltonian \hat{H}_0 obeys additional time-reversal symmetry $T' = \sigma_z T$ that is equivalent to the introduction of an unessential phase into the ψ_2 component $\psi_2 \rightarrow e^{i\pi}\psi_2$ that is decoupled from the ψ_1



FIG. 2. Energy of linear states versus Bloch momentum k in a truncated Zeeman lattice with zigzag-bearded edges for different SOC strengths β_x and β_y . Black curves correspond to bulk modes, while red (green) curves correspond to edge states residing on the left zigzag (right bearded) edge. The lattice parameters are $\rho = 8$, a = 1.4, and d = 0.5. The Chern numbers $C_{1,2}$ of the two lowest bands of the respective bulk lattice are shown in the panels. Here and in all other figures all quantities are plotted in dimensionless units.

component in the absence of SOC. Since $[\mathcal{T}', \beta_y \sigma_x \hat{k}_y] = 0$, the component $\beta_y \sigma_x \hat{k}_y$ does not break time-reversal symmetry \mathcal{T}' and cannot lead to topological effects.

The symmetry considerations for infinite lattice have direct implications for the formation of edge states in the truncated lattice as illustrated in Fig. 2. Without SOC, $\beta_x = \beta_y = 0$, one observes two bands touching in Dirac points at k = K/3and k = 2K/3 that are remnants of the Dirac points of the bulk lattice. Two nontopological edge states arise at zigzag (red curve) and bearded (green curve) edges. When $\beta_x = 0$ the SOC component $\sim \beta_{y}$ opens a gap that is nontopological in accordance with the above considerations. In this gap one encounters only one nontopological edge state (it does not connect two bands; see second panel of Fig. 2). The inclusion of a weak SOC component $\sim \beta_x$ cannot immediately lead to the appearance of topological edge states: since a nontopological gap already exists, it should first close under the action of the β_x component (this occurs around $\beta_x = 0.85$, third panel of Fig. 2) and then reopen in the form of a topological gap at $\beta_x > 0.85$ (fourth panel, Fig. 2), where two topological modes connecting different bands appear. The topological phase transition is confirmed by Chern numbers of the respective bands [35]: $C_n = 1/(2\pi) \int_{BZ} [\partial_x A_{ny}(\mathbf{k}) - \partial_y A_{nx}(\mathbf{k})] d^2k$, where $A_{n\alpha} = i \langle \boldsymbol{\psi}_{n\mathbf{k}} | \partial_{k_{\alpha}} | \boldsymbol{\psi}_{n\mathbf{k}} \rangle$ is the Berry connection, $\boldsymbol{\psi}_{n\mathbf{k}}$ is the Bloch function of the *n*th band, $\alpha = x$, y, and the integral is over the first Brillouin zone [36]. Before the gap closing in Fig. 2, the lowest bands have equal Chern numbers $C_{1,2} = 0$. After gap reopening the bands acquire Chern numbers $C_1 = -1$ and $C_2 = 1$. According to bulk-edge correspondence this corresponds to a single topological edge state in the truncated lattice.

A different scenario is observed when β_x is large and one gradually increases β_y contribution. At $\beta_y = 0$ no gap exists in the spectrum even if $\beta_x \sim 1$, as was explained above. By adding even a weak $\beta_y \sigma_x \hat{k}_y$ term into the Hamiltonian, one breaks the time-reversal symmetry \mathcal{T} and, hence, a topological gap appears (the Hamiltonian with $\beta_x > 0$ does not obey \mathcal{T}' symmetry). The width of this gap monotonically increases with β_y , and unidirectional edge states exist at different edges, as shown in the fifth and sixth panels of Fig. 2.

These results imply that both β_x and β_y SOC components are required for the existence of topological states. The largest topological gap was achieved for pure Rashba coupling, when $\beta_x = \beta_y$ (sixth panel of Fig. 2). Results presented in this figure constitute the central finding of this Rapid Communication: the emergence of topological edge states due to the interplay of SOC and the Zeeman lattice, induced by the inhomogeneous magnetic field.

Further we concentrate on the case of Rashba coupling $\beta_x = \beta_y$. The sixth panel of Fig. 2 indicates that edge states corresponding to zigzag and bearded edges may coexist for small intervals of Bloch momentum, but in general, they occupy different domains in *k* [38]. Representative examples of topological edge states are shown in Fig. 1. The best localization of the edge estate is achieved when energy ε falls into the center of the topological gap. When energy approaches the lower or upper allowed bands, the mode strongly expands toward the center of the lattice.

Dispersive properties of the edge states are summarized in Fig. 3 where k dependencies of derivatives $\varepsilon' \equiv \partial \varepsilon / \partial k$ and $\varepsilon'' \equiv \partial \varepsilon / \partial k$ are shown. A peculiarity of our system is that edge states are not necessarily moving: the group velocity $\varepsilon' \equiv \partial \varepsilon / \partial k$ can be zero for the zigzag edge modes and can even have two zeros for the bearded edge modes [Fig. 3(a)]. For quantum spin Hall states such zero group velocity modes on conventional graphene lattices were observed in [39]. Thus, by changing the Bloch momentum one can control the direction of the surface current. An unusual situation is possible in the vicinity of k = K/2, where the states at the opposite edges move in the same direction (the red and green curves closely approach). Also, Fig. 3(b) suggests that second-order dispersion may vanish for some edge states. When a broad envelope is superimposed on such a state, the latter moves along the interface over hundreds of lattice sites, representing a linear quasi-non-dispersing wave packet.



FIG. 3. Group velocity ε' (a) and dispersion ε'' (b) vs Bloch momentum k for edge states from zigzag (red) and bearded (green) edges at $\beta_x = \beta_y = 1.5$. Dashed lines indicate $\varepsilon' = 0$ or $\varepsilon'' = 0$ level.



FIG. 4. Honeycomb lattice with zigzag edges and $|\psi_1|$ distributions at different moments of time illustrating circulation of the edge state with broad envelope at k = 0.40K, $\beta_x = \beta_y = 1.5$.

The example of a topologically protected spinor surface current is shown in Fig. 4. We consider a triangular lattice with zigzag edges, with each edge containing 30 periods of the honeycomb structure. The initial state $\psi(x, y) = \phi(x, y)e^{iky}e^{-y^2/w^2}$ at t = 0 was prepared using edge state ϕ corresponding to Bloch momentum k = 0.4K with envelope of width w = 20. Such a state can traverse several corners of triangular lattice returning to its original location. Although the triangular shape of the lattice is less favorable for persistent circular currents because of relatively strong scattering of edge states into the bulk modes at the corners, an almost complete round trip of the wave packet still occurs.

In a SO-BEC, where atoms in a Zeeman lattice experience two-body interactions, the two-component order parameter ψ is governed by the coupled Gross-Pitaevski equations:

$$i\frac{\partial\boldsymbol{\psi}}{\partial t} = \frac{1}{2}\left(\frac{1}{i}\nabla + \mathbf{A}\right)^2\boldsymbol{\psi} + \hat{U}(\mathbf{r})\boldsymbol{\psi} + g(\boldsymbol{\psi}^{\dagger}\boldsymbol{\psi})\boldsymbol{\psi}.$$
 (1)

Here $\mathbf{A} = -\mathbf{i}\beta_x\sigma_y + \mathbf{j}\beta_y\sigma_x$ is the non-Abelian gauge potential and g characterizes inter- and intraspecies interactions, which are considered equal [25]. The sign of g coincides with the sign of scattering length for two-body interactions and is considered relatively small, such that the BEC is in the superfluid phase for the chosen Zeeman lattice. Here we are interested in the impact of the nonlinearity on the



FIG. 5. Amplitudes of the components (a) and number of atoms per y period (b) of nonlinear edge states at the zigzag edge versus chemical potential μ for $\beta_x = \beta_y = 1.5$. Black (red) curves correspond to g = -1 (g = +1). Vertical dashed lines indicate borders of the topological gap. Red dots correspond to linear edge states.



FIG. 6. Evolution of the perturbed nonlinear edge state from zigzag edge at $\mu = -3.43$, k = 0.40K, $\beta_x = \beta_y = 1.5$, and g = -1.

properties of topological edge states, considered above for the linear case. Topological states now are searched as *nonlinear* Bloch modes parametrized by the chemical potential μ and Bloch momentum $k: \psi(x, y) = e^{iky-i\mu t}\phi(x, y)$. Bifurcations of such modes from the linear edge states with $\mu = \varepsilon$ at the zigzag boundary are depicted in Fig. 5, where we plot the peak amplitudes $a_{1,2}$ of the spinor components and the number of atoms $N = \int_{-\infty}^{\infty} dx \int_{0}^{T} dy \psi^{\dagger} \psi$ per *y* period, as functions of the chemical potential.

We considered positive, g > 0, and negative, g < 0, scattering lengths and found that amplitudes and norms of the nonlinear edge states monotonically increase toward the gap edges. They vanish in the point where they bifurcate from the linear edge state (red dots in Fig. 5). In all cases the first spinor component dominates. Both attractive and repulsive interactions lead to delocalization of the nonlinear edge states and their strong expansion into bulk of the lattice when the chemical potential approaches one of the topological gap edges.

Rigorously, nonlinear edge states in Zeeman lattices were found to be unstable. However, these instabilities may develop at very large evolution times, therefore in practice the states may be seen as metastable. For a topological edge state, instability is determined by the number of atoms N and by the point of bifurcation of a nonlinear mode from the linear limit (the red dot in Fig. 5). The typical timescale of instability is larger for nonlinear modes bifurcating from the center of the gap and it decreases with N, when the chemical potential approaches a gap edge ([36]). For g < 0 and g > 0instabilities are qualitatively different, as illustrated in Figs. 6 and 7, respectively. For the selected value of k = 0.4K the effective mass $\sim 1/\varepsilon''$ along the y axis is negative. For g < 0this implies that no modulational instability can develop in



FIG. 7. Evolution of the perturbed nonlinear edge state from zigzag edge at $\mu = -3.34$, k = 0.40K, $\beta_x = \beta_y = 1.5$, and g = +1.

the y direction along the interface. Indeed, no splitting into regular fragments along the edge is observed in Fig. 6 and instability shows up as a dispersion of the edge state into the bulk. In the case of g > 0 the development of modulational instability is possible for negative effective mass and one can clearly see fragmentation of the wave into a periodic pattern in Fig. 7. This can be considered as a precursor to the formation of edge solitons.

Returning to the example of topologically protected spinor surface current illustrated in Fig. 4 for a triangular structure, we verified that one round trip, similar to the one shown in Fig. 4, still occurs for moderate attractive and repulsive nonlinearities that only lead to an increase of the bulk radiation [36].

Summarizing, we have revealed the existence of topological edge states in atomic systems with SOC loaded in a honeycomb Zeeman lattice. We explored a wide range of parameters characterizing orthogonal components of the SOC and found that they play fundamentally different roles in the gap opening and edge state formation. The obtained states demonstrate persistent spinor surface currents in finite-size lattices. When two-body interactions are included and the atomic system becomes a spin-orbit-coupled BEC, nonlinear metastable topological edge states can exist.

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- F. D. M. Haldane, Model for a Quantum Hall Effect Without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly," Phys. Rev. Lett. 61, 2015 (1988).
- [2] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, and A. K. Geim, The electronic properties of graphene, Rev. Mod. Phys. 81, 109 (2009).
- [3] C. L. Kane and E. J. Mele, Topological Order and the Quantum Spin Hall Effect in Graphene, Phys. Rev. Lett. 95, 226801 (2005).
- [4] M. Z. Hasan and C. L. Kane, *Colloquium*: Topological insulators, Rev. Mod. Phys. 82, 3045 (2010).
- [5] X.-L. Qi and S.-C. Zhang, Topological insulators and superconductors, Rev. Mod. Phys. 83, 1057 (2011).
- [6] C. He, X. Ni, H. Ge, X. C. Sun, Y. B. Chen, M. H. Lu, X. P. Liu, and Y. F. Chen, Acoustic topological insulator and robust one-way sound transport, Nat. Phys. 12, 1124 (2016).
- [7] R. Susstrunk and S. D. Huber, Observation of phononic helical edge states in a mechanical topological insulator, Science 349, 47 (2015).
- [8] L. Lu, J. D. Joannopoulos, and M. Soljacic, Topological photonics, Nat. Photon. 8, 821 (2014).
- [9] F. D. M. Haldane and S. Raghu, Possible Realization of Directional Optical Waveguides in Photonic Crystals with Broken Time-Reversal Symmetry, Phys. Rev. Lett. 100, 013904 (2008).
- [10] S. Raghu and F. D. M. Haldane, Analogs of quantum-Halleffect edge states in photonic crystals, Phys. Rev. A 78, 033834 (2008).
- [11] Z. Wang, Y. Chong, J. D. Joannopoulos, and M. Soljačić, Observation of unidirectional backscattering-immune topological electromagnetic states, Nature (London) 461, 772 (2009).
- [12] N. H. Lindner, G. Refael, and V. Galitski, Floquet topological insulator in semiconductor quantum wells, Nat. Phys. 7, 490 (2011).
- [13] M. Hafezi, E. A. Demler, M. D. Lukin, and J. M. Taylor, Robust optical delay lines with topological protection, Nat. Phys. 7, 907 (2011).

- [14] O. Umucalilar and I. Carusotto, Fractional Quantum Hall States of Photons in An Array of Dissipative Coupled Cavities, Phys. Rev. Lett. **108**, 206809 (2012).
- [15] A. B. Khanikaev, S. H. Mousavi, W.-K. Tse, M. Kargarian, A. H. MacDonald, and G. Shvets, Photonic topological insulators, Nat. Mater. 12, 233 (2012).
- [16] M. C. Rechtsman, J. M. Zeuner, Y. Plotnik, Y. Lumer, D. Podolsky, F. Dreisow, S. Nolte, M. Segev, and A. Szameit, Photonic Floquet topological insulators, Nature (London) 496, 196 (2013).
- [17] L. J. Maczewsky, J. M. Zeuner, S. Nolte, and A. Szameit, Observation of photonic anomalous Floquet topological insulators, Nat. Commun. 8, 13756 (2017).
- [18] S. Mukherjee, A. Spracklen, M. Valiente, E. Andersson, P. Öhberg, N. Goldman, and R. R. Thomson, Experimental observation of anomalous topological edge modes in a slowly driven photonic lattice, Nat. Commun. 8, 13918 (2017).
- [19] C.-E. Bardyn, T. Karzig, G. Refael, and T. C. H. Liew, Chiral Bogoliubov excitations in nonlinear bosonic systems, Phys. Rev. B 93, 020502(R) (2016).
- [20] V. Nalitov, D. D. Solnyshkov, and G. Malpuech, Polariton Z Topological Insulator, Phys. Rev. Lett. 114, 116401 (2015).
- [21] T. Karzig, C.-E. Bardyn, N. H. Lindner, and G. Refael, Topological Polaritons, Phys. Rev. X 5, 031001 (2015).
- [22] Y. V. Kartashov and D. V. Skryabin, Modulational instability and solitary waves in polariton topological insulators, Optica 3, 1228 (2016); C. Li, F. Ye, X. Chen, Y. V. Kartashov, A. Ferrando, L. Torner, and D. V. Skryabin, Lieb polariton topological insulators, Phys. Rev. B 97, 081103(R) (2018).
- [23] M. C. Beeler, R. A. Williams, K. Jiménez-García, L. J. LeBlanc, A. R. Perry, and I. B. Spielman, The spin Hall effect in a quantum gas, Nature (London) 498, 201 (2013).
- [24] C. J. Kennedy, G. A. Siviloglou, H. Miyake, W. C. Burton, and W. Ketterle, Spin-Orbit Coupling and Quantum Spin Hall Effect for Neutral Atoms without Spin Flips, Phys. Rev. Lett. 111, 225301 (2013).

- [25] Y.-J. Lin, K. Jiménez-García, and I. B. Spielman, Spin-orbitcoupled Bose-Einstein condensates, Nature (London) 471, 83 (2011).
- [26] V. Galitski and I. B. Spielman, Spin-orbit coupling in quantum gases, Nature (London) 494, 49 (2013).
- [27] J. Ruseckas, G. Juzeliūnas, P. Öhberg, and M. Fleischhauer, Non-Abelian Gauge Potentials for Ultracold Atoms with Degenerate Dark States, Phys. Rev. Lett. 95, 010404 (2005); J. Dalibard, F. Gerbier, G. Juzeliunas, and P. Öhberg, *Colloquium*: Artificial gauge potentials for neutral atoms, Rev. Mod. Phys. 83, 1523 (2011).
- [28] N. R. Cooper, Optical Flux Lattices for Ultracold Atomic Gases, Phys. Rev. Lett. 106, 175301 (2011).
- [29] K. Jiménez-García, L. J. LeBlanc, R. A. Williams, M. C. Beeler, A. R. Perry, and I. B. Spielman, Peierls Substitution in An Engineered Lattice Potential, Phys. Rev. Lett. **108**, 225303 (2012).
- [30] B. Galilo, D. K. K. Lee, and R. Barnett, Topological Edge-State Manifestation of Interacting 2D Condensed Boson-Lattice Systems in A Harmonic Trap, Phys. Rev. Lett. 119, 203204 (2017).
- [31] E. I. Rashba, Properties of semiconductors with an extremum loop: I. Cyclotron and combinational resonance in a magnetic field perpendicular to the plane of the loop, Sov. Phys. Solid State 2, 1109 (1960).
- [32] J. Radić, A. Di Ciolo, K. Sun, and V. Galitski, Exotic Quantum Spin Models in Spin-Orbit-Coupled Mott Insulators, Phys. Rev. Lett. **109**, 085303 (2012); W. S. Cole, S. Zhang, A. Paramekanti, and N. Trivedi, Bose-Hubbard Models with

Synthetic Spin-Orbit Coupling: Mott Insulators, Spin Textures, and Superfluidity, *ibid.* **109**, 085302 (2012).

- [33] M. Zarea and N. Sandler, Rashba spin-orbit interaction in graphene and zigzag nanoribbons, Phys. Rev. B 79, 165442 (2009); P. Rakyta, A. Kormányos, and J. Cserti, Trigonal warping and anisotropic band splitting in monolayer graphene due to Rashba spin-orbit coupling, *ibid.* 82, 113405 (2010).
- [34] M. J. Ablowitz and Yi Zhu, Evolution of Bloch-mode envelopes in two-dimensional generalized honeycomb lattices, Phys. Rev. A 82, 013840 (2010).
- [35] J. Zak, Berry's Phase for Energy Bands in Solids, Phys. Rev. Lett. 62, 2747 (1989); D. Xiao, M.-C. Chang, and Q. Niu, Berry phase effects on electronic properties, Rev. Mod. Phys. 82, 1959 (2010).
- [36] See Supplemental Material at http://link.aps.org/supplemental/ 10.1103/PhysRevA.98.061601 for details of transformations of the band structure, the Chern number computed using the algorithm of [37], as well as the effect of the interatomic interactions of the evolution of the topological edge state.
- [37] T. Fukui, Y. Hatsugai, and H. Suzuki, Chern numbers in discretized Brillouin zone: Efficient method of computing (spin) Hall conductances, J. Phys. Soc. Jpn. 74, 1674 (2005).
- [38] M. J. Ablowitz, C. W. Curtis, and Yi Zhu, Localized nonlinear edge states in honeycomb lattices, Phys. Rev. A 88, 013850 (2013).
- [39] Y. Yang, Z. Xu, L. Sheng, B. Wang, D. Y. Xing, and D. N. Sheng, Time-Reversal-symmetry-Broken Quantum Spin Hall Effect, Phys. Rev. Lett. **107**, 066602 (2011).